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Analytic method for solving the nonlinear Schrödinger equation describing pulse propagation in dispersive optic fibres

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Abstract. We give a method for obtaining new exact solutions of the nonlinear Schrödinger equation describing pulse propagation in optical fibres for both the anomalous and the normal dispersion regime. The method is based on the construction of a certain complete integrable finite-dimensional dynamical system whose solution determines the exact solutions of the nonlinear Schrödinger equation. By using the phase diagrams associated with the corresponding nonlinear differential equations we classify all the obtained solutions into one of the following categories: bright or dark solitary waves, bright or dark soliton solutions, rational (algebraic) bright or dark solitons, regular or singular periodic waves and stationary solutions.

We give a set of particular solutions which describe the periodic wave patterns that are generated by the temporal self-phase modulation instability, the periodic evolution of bright solitons on a continuous wave background and the collision of two dark waves with equal amplitudes.

1. Introduction

Optical solitons in fibres are pulses that propagate without any change in pulse shape or intensity. Because of their remarkable stability properties, optical solitons are now at the centre of an active research field of nonlinear wave propagation in optical fibres. This research field started with the result [1–2] that under appropriate combinations of pulse shape and intensity, the effects of the intensity-dependent refractive index of the fibre exactly compensate for the pulse-spreading effects of group velocity dispersion. For negative group velocity dispersion or anomalous dispersion regime ($\partial^2k/\partial\omega^2 < 0$) which occurs in typical single-mode silica based fibres for wavelengths $\lambda > 1.27$ μm the fundamental soliton is called a bright pulse [1] and the propagation of these bright solitons has been studied intensively and verified experimentally [3]. For positive group velocity dispersion or normal dispersion regime ($\partial^2k/\partial\omega^2 > 0$) the theory [2] and numerical simulations [4–5] predict that the solitons are dark pulses (i.e., a dip occurs at the centre of the pulse). The generation of dark solitons in single-mode optical fibres was also demonstrated [6–8]. Recently a new soliton transmission technique which makes positive use of the existence of slight fibre loss, called dynamic soliton communication, was used to send optical solitons over long distances [9]. It was demonstrated that digitally-coded optical solitons at a bit rate of 20 Gbit s^{-1} can be successfully transmitted over 1020 km using erbium-doped fibre amplifiers [10]. We mention also the works of several very active research groups in the field of the theory of pulse propagation in optical fibres in both the picosecond and the femtosecond regime [11–32].

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The propagation of optical pulses in mono-mode optical fibres exhibiting Kerr-law nonlinearities is described well by the dimensionless nonlinear Schrödinger equation (NLSE):

$$\mathrm{i}\psi_x + \epsilon\psi_{tt} + 2|\psi|^2\psi = 0 \tag{1.1}$$

where ψ represents a normalized complex amplitude of the pulse envelope, x is a normalized distance along the fibre, t is the normalized retarded time (we employ a frame of reference moving with the pulse at the group velocity v_g), $\epsilon = 1$ corresponds to the anomalous dispersion region where bright solitons can exist and $\epsilon = -1$ corresponds to the normal dispersion region where dark solitons occur.

The NLSE is one of the complete integrable nonlinear partial differential equations and the solutions may be obtained by different methods, e.g., by using the inverse scattering method [33–39] or the Lie groups theory [40–42]. Another way of obtaining solutions of NLSE is the Darboux transformation method [43]. We mention also the work on inverse scattering transform perturbation theory for soliton propagation and the extended first- and second-order perturbation expansion for soliton propagation in optical fibres [44].

The two-parameter one-soliton solution of the NLSE (1.1) can be written as [33]:

$$\psi_{\mathrm{b}}(x,t) = q\,\mathrm{sech}[q(t+2\kappa x)]\exp[\mathrm{i}(q^2-\kappa^2)x - \mathrm{i}\kappa t] \qquad \epsilon = 1 \tag{1.2}$$

$$\psi_{\mathrm{d}}(x,t) = q\tanh[q(t+2\kappa x)]\exp[\mathrm{i}(2q^2+\kappa^2)x + \mathrm{i}\kappa t] \qquad \epsilon = -1 \tag{1.3}$$

where q is a form factor that determine the pulse amplitude and width, κ is the frequency shift of the soliton and the subscripts b and d denote bright and dark pulses, respectively.

The canonical single-soliton solutions of the NLSE (1.1) are given by [1–2]:

$$\psi'_{\mathrm{b}}(x,t) = \mathrm{sech}\,t\exp(\mathrm{i}x) \qquad \epsilon = 1 \tag{1.4}$$

$$\psi'_{\mathrm{d}}(x,t) = \tanh t\exp(2\mathrm{i}x) \qquad \epsilon = -1. \tag{1.5}$$

An important scaling relation holds for the NLSE (1.1). If $\psi'(x,t)$ is a solution of this equation then

$$\psi(x,t) = q\psi'(q^2x, qt) \tag{1.6}$$

is also a solution, where q is an arbitrary scaling factor:

$$\psi_{\mathrm{b}}(x,t) = q\,\mathrm{sech}(qt)\exp(\mathrm{i}q^2x) \qquad \epsilon = 1 \tag{1.7}$$

$$\psi_{\mathrm{d}}(x,t) = q\tanh(qt)\exp(2\mathrm{i}q^2x) \qquad \epsilon = -1. \tag{1.8}$$

Recently a new method of obtaining exact solutions of the NLSE (1.1) for describing pulse propagation in optical fibres in the anomalous dispersion regime ($\epsilon = 1$) was given [45]. This method came from the observation [46] that the one-soliton solutions and the periodic solutions which describe the development of the self-phase modulation instability [47] belong to a large class of complex solutions $\psi(x,t) = u(x,t) + \mathrm{i}v(x,t)$, the so-called first-order solutions of the NLSE (1.1) for which a linear relationship

$$u(x,t) - a_0(x)v(x,t) - b_0(x) = 0 \tag{1.9}$$

holds between the real part $u(x,t)$ and the imaginary part $v(x,t)$ of the complex function $\psi(x,t)$, where the coefficients a_0 and b_0 depend only on the spatial coordinate x (the normalized distance along the fibre). We note that the two-soliton, or more generally, the n-soliton solutions ($n \geqslant 2$) and the periodic solutions with more than one period in the time variable t do not belong to this set of solutions of the NLSE (1.1) for which the linear relationship (1.9) holds [46].

The method developed in [45] is essentially the construction of a certain system of ordinary differential equations the solutions of which determine the solutions of NLSE (1.1).

In section 2 we shall present the method which allows us to obtain new exact solutions of NLSE (1.1) for $\epsilon = \pm 1$. By using the linear relationship (1.9) between the unknown functions $u(x, t)$ and $v(x, t)$ we will construct a certain dynamical system, the solution of which determines the exact solution of NLSE (1.1) with $\epsilon = \pm 1$. In the general case we obtain a three-parameter family of solutions of NLSE (1.1) which are expressed in terms of the Jacobi elliptic functions and the incomplete elliptic integral of the third kind [48].

In section 3 we list a compendium of solutions that are relevant to equation (1.1) with $\epsilon = \pm 1$. In the last section we briefly present our conclusions.

By using the phase diagrams associated with the corresponding nonlinear differential equations we classify all the obtained analytical solutions into one of the following categories: bright or dark solitary waves, bright or dark soliton solutions, rational (algebraic) bright or dark solitons, regular or singular periodic waves and stationary solutions. This simple geometrical way to classify the solutions of the corresponding nonlinear differential equations associated with the NLSE (1.1) is more suggestive than the analysis made in [45] on the general three-parameter family of solutions. In addition, our paper contains a comprehensive analysis of the so-called first-order solutions of the NLSE (1.1) for which the linear relationship (1.9) holds for both the positive and the negative dispersion regimes.

2. The description of the method

We introduce new unknown functions $Q(x, t)$, $\delta(x)$ and $\varphi(x)$ through the following relations [45]:

$$a_0(x) = \operatorname{cotan} \varphi(x) \tag{2.1}$$

$$b_0(x) = -\frac{\delta(x)}{\sin \varphi(x)} \tag{2.2}$$

$$u(x, t) = Q(x, t) \cos \varphi(x) - \delta(x) \sin \varphi(x) \tag{2.3}$$

such that we have the following representation for the unknown function $\psi(x, t)$:

$$\psi(x, t) = [Q(x, t) + \mathrm{i}\delta(x)] \exp[\mathrm{i}\varphi(x)]. \tag{2.4}$$

By introducing (2.4) in the NLSE (1.1) and taking the real and imaginary parts we are left with the following system of differential equations:

$$\epsilon Q_{tt} - \delta_x - \varphi_x Q + 2\delta^2 Q + 2Q^3 = 0 \tag{2.5}$$

$$Q_x - \varphi_x \delta + 2\delta Q^2 + 2\delta^3 = 0. \tag{2.6}$$

Here the differential equation (2.5) has a first integral:

$$\epsilon Q_t^2 + Q^4 + (2\delta^2 - \varphi_x) Q^2 - 2\delta_x Q = \epsilon h(x) \tag{2.7}$$

where $h(x)$ is a function which depends only on the spatial variable x.

The condition of compatibility of the system of differential equations (2.5) and (2.6), i.e., $Q_{xt} = Q_{tx}$ gives the following system of three ordinary differential equations for the unknown functions $\varphi(x)$, $\delta(x)$, and $h(x)$:

$$\varphi_{xx} + 8\delta\delta_x = 0 \tag{2.8}$$

$$\delta_{xx} + 4\epsilon\delta h + \delta\varphi_x^2 - 4\delta^3\varphi_x + 4\delta^5 = 0 \tag{2.9}$$

$$\epsilon h_x + 2\delta\delta_x\varphi_x - 4\delta^3\delta_x = 0. \tag{2.10}$$

The dynamical system (2.8)–(2.10) corresponding to (1.1) with $\epsilon = \pm 1$ has the following three first integrals:

$$\varphi_x + 4\delta^2 = W \tag{2.11}$$

$$\epsilon h + W\delta^2 - 3\delta^4 = \epsilon H \tag{2.12}$$

$$\delta_x^2 + (W^2 + 4\epsilon H)\delta^2 - 8W\delta^4 + 16\delta^6 = G. \tag{2.13}$$

Next with the help of the substitution $z(x) = \delta^2(x)$ we obtain:

$$z_x^2 = -64z^4 + 32Wz^3 - 4(W^2 + 4\epsilon H)z^2 + 4Gz. \tag{2.14}$$

Now let $\alpha_0 = 0$, α_1, α_2, α_3 be the roots of the polynomial on the right-hand side of (2.14). These roots are connected with the prime integrals W, H and G via the Viète relations:

$$W = 2(\alpha_1 + \alpha_2 + \alpha_3) \tag{2.15}$$

$$\epsilon H = 2\alpha_1\alpha_2 + 2\alpha_2\alpha_3 + 2\alpha_3\alpha_1 - \alpha_1^2 - \alpha_2^2 - \alpha_3^2 \tag{2.16}$$

$$G = 16\alpha_1\alpha_2\alpha_3. \tag{2.17}$$

Next the equations (2.7) and (2.14) become, respectively:

$$Q_t^2 = -\epsilon P_1(Q) \cdot P_2(Q) \tag{2.18}$$

$$z_x^2 = -64z(z - \alpha_1)(z - \alpha_2)(z - \alpha_3). \tag{2.19}$$

The polynomials on the right-hand side of (2.18) have the form (see [49] p 24):

$$P_{1,2}(Q) = Q^2 \pm 2(\alpha_3 - z)^{1/2}Q + \alpha_3 - \alpha_1 - \alpha_2 + z \pm 2[(\alpha_1 - z)(\alpha_2 - z)]^{1/2}. \tag{2.20}$$

The simplest solution of (2.19) is the constant function $z = 0$ which gives the stationary solution of (1.1). We note that the function z cannot be one of the roots α_i ($i = 1, 2, 3$) because in this case the equation (2.6) cannot be fulfilled. Similarly, (2.18) has the solutions $Q = Q_i(x)$, $i = 1, \ldots, 4$, where $Q_i(x)$ are the roots of the polynomial $P_1(Q) \cdot P_2(Q)$. For these particular solutions the function ψ depends only on the spatial variable x. We observe also from (2.19) and (2.11) that the functions $z(x)$ and $\varphi(x)$ have the same expressions for both $\epsilon = 1$ and $\epsilon = -1$.

From (2.19) it is easy to see that at least one of the roots α_i is positive and in the following we suppose that $\alpha_3 \geqslant 0$. Because the functions $Q(x,t)$ and $z(x)$ are real we have two distinct situations: a) α_1, α_2, α_3 are real numbers and b) $\alpha_3 \geqslant 0$, α_1 and α_2 are complex conjugate numbers ($\alpha_1 = \alpha_2^* = \rho + i\eta$). In the case a) the discriminants $D_{1,2}$ of the polynomials $P_{1,2}(Q)$ are:

$$D_{1,2} = \begin{cases} -[(z-\alpha_1)^{1/2} \pm (z-\alpha_2)^{1/2}]^2 & \text{for } \alpha_2 \leqslant z \leqslant \alpha_3 \\ [(\alpha_1 - z)^{1/2} \mp (\alpha_2 - z)^{1/2}]^2 & \text{for } 0 \leqslant z \leqslant \alpha_1 \end{cases} \tag{2.21}$$

where $D_{1,2}$ corresponds to $P_1(Q)$ and $P_2(Q)$, respectively and in the case b) the discriminants $D_{1,2}$ are:

$$D_{1,2} = 2(\rho - z) \mp 2[(\rho - z)^2 + \eta^2]^{1/2}. \tag{2.22}$$

By using the expressions (2.21) and (2.22) for the discriminants $D_{1,2}$ we can choose different kinds of root degeneracies of the polynomial on the right-hand side of (2.18) by selecting the constants α_i ($i = 1, 2, 3$).

Finally knowing the functions $Q(x,t)$, $z(x)$ and $\varphi(x)$ we can write the solution $\psi(x,t)$ of the NLSE (1.1):

$$\psi(x,t) = \{Q(x,t) + i[z(x)]^{1/2}\} \exp[i\varphi(x)]. \tag{2.23}$$

3. Analytic solutions

The relation between Q_t^2 and Q given by (2.18) can be illustrated by the phase diagrams of figures 1 and 2 corresponding to $\epsilon = -1$ and $\epsilon = 1$, respectively.

In the following we will list the solutions of equation (1.1) with $\epsilon = \pm 1$ for different kinds of root degeneracies of the polynomial on the right-hand side of (2.18). They are all referred to phase diagrams of figures 1 and 2.

3.1. $Q_1 = Q_2 = Q_3 = Q_4 = 0$

For this situation, we have the singular stationary solution which exists only for $\epsilon = -1$ because Q is a real function:

$$Q = \frac{1}{t}. \tag{3.1}$$

This case is realized when $\alpha_1 = \alpha_2 = \alpha_3 = 0$ and $z = 0$ (see figure 1(*a*)). Thus we obtain the following solution of NLSE (1.1) with $\epsilon = -1$:

$$\psi(x,t) = \frac{1}{t} e^{i\varphi_0} \tag{3.2}$$

where φ_0 is a constant phase.

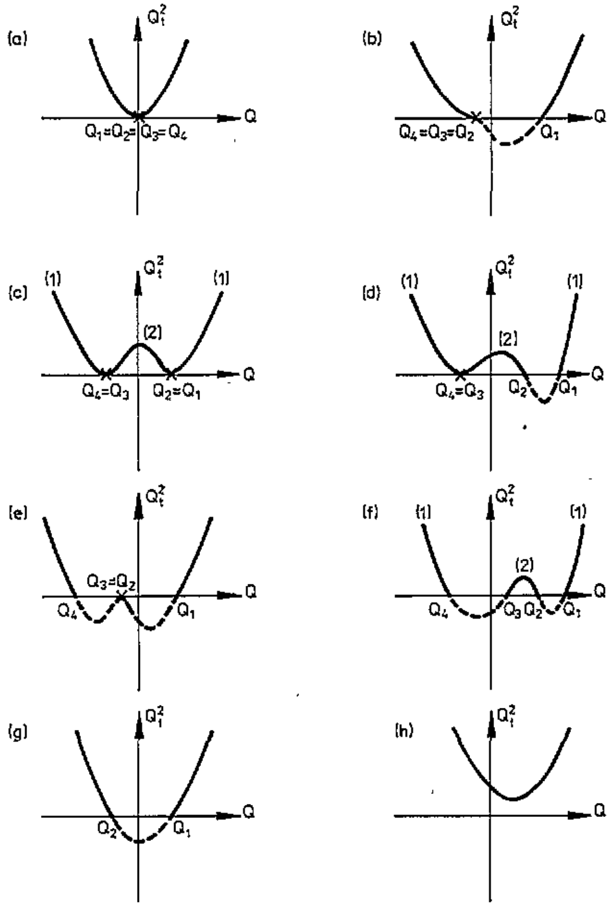

Figure 1. Phase diagrams for (2.18) with $\epsilon = -1$ for different kinds of roots degeneracies (the crosses denote a multiple zero of Q_t^2). (*a*) $\alpha_1 = \alpha_2 = \alpha_3 = 0$ and $z = 0$, (*b*) $\alpha_1 = \alpha_2 = \alpha_3 > 0$ and $0 \leqslant z \leqslant \alpha_1$, (*c*) $\alpha_1 = \alpha_2 = 0$, $\alpha_3 > 0$ and $z = 0$, (*d*) $\alpha_1 = \alpha_2 > 0$, $\alpha_3 > \alpha_1$ and $0 \leqslant z \leqslant \alpha_1$, (*e*) $\alpha_2 = \alpha_3 > 0$ and $0 \leqslant z \leqslant \alpha_1 < \alpha_2$, (*f*) $\alpha_1 < \alpha_2 < \alpha_3$ and $0 \leqslant z \leqslant \alpha_1$, (*g*) $\alpha_1 = \alpha_2^*$ and $0 \leqslant z \leqslant \alpha_3$, (*h*) $\alpha_1 < \alpha_2 \leqslant z \leqslant \alpha_3$ and $z \geqslant 0$.

3.2. $Q_4 = Q_3 = Q_2 < Q_1$

Here we have analytic solutions for both anomalous ($\epsilon = 1$) and normal ($\epsilon = -1$) dispersion regimes:

$$Q = (\alpha_1 - z)^{1/2} \frac{3 - 4\epsilon(\alpha_1 - z)t^2}{1 + 4\epsilon(\alpha_1 - z)t^2} \tag{3.3}$$

where $Q(x, t = 0) = Q_1 = 3(\alpha_1 - z)^{1/2}$, $Q_2 = Q_3 = Q_4 = -(\alpha_1 - z)^{1/2}$.

For $\epsilon = 1$ the solution (3.3) represents the bright rational (algebraic) solitary wave and for $\epsilon = -1$ the singular rational (algebraic) solution.

This case is obtained for the choice $\alpha_1 = \alpha_2 = \alpha_3 > 0$ and $0 \leqslant z \leqslant \alpha_1$ (see figures 1(*b*) and 2(*a*)). Thus we find the following expressions for the functions $z(x)$ and $\varphi(x)$:

$$z(x) = \frac{16\alpha_1^3 x^2}{1 + 16\alpha_1^2 x^2} \tag{3.4}$$

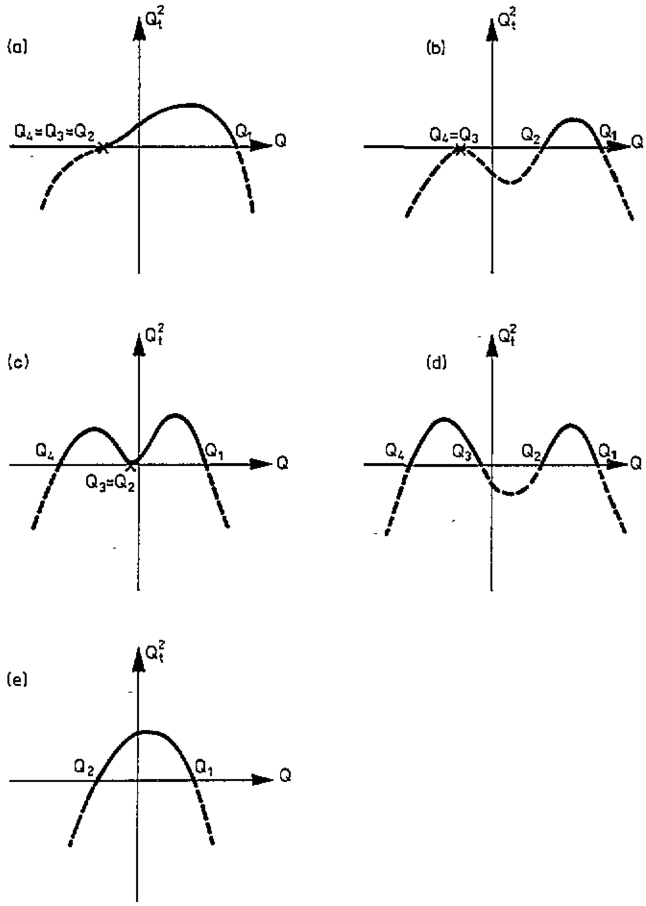

Figure 2. Phase diagrams for (2.18) with $\epsilon = 1$ for different kinds of roots degeneracies (the crosses denote a multiple zero of Q_t^2). (*a*) $\alpha_1 = \alpha_2 = \alpha_3 > 0$ and $0 \leqslant z \leqslant \alpha_1$, (*b*) $\alpha_1 = \alpha_2 > 0$, $\alpha_3 > \alpha_1$ and $0 \leqslant z \leqslant \alpha_1$, (*c*) $\alpha_2 = \alpha_3 > 0$ and $0 \leqslant z \leqslant \alpha_1 < \alpha_2$, (*d*) $\alpha_1 < \alpha_2 < \alpha_3$ and $0 \leqslant z \leqslant \alpha_1$, (*e*) $\alpha_1 = \alpha_2^*$ and $0 \leqslant z \leqslant \alpha_3$.

$$\varphi(x) = 2\alpha_1 x + \arctan(4\alpha_1 x). \tag{3.5}$$

We note that the solutions of the NLSE (1.1) obtained by using this direct method form a three-parameter family. Let $\psi(x, t)$ be the solution of the NLSE (1.1) corresponding to the three real parameters α_i ($i = 1, 2, 3$), where at least one is positive ($\alpha_3 \geqslant 0$) and let $\psi'(x, t)$ be the solution of (1.1) corresponding to the two parameters $a_1 = \alpha_1/2\alpha_3$ and $a_2 = \alpha_2/2\alpha_3$ ($a_3 = \frac{1}{2}$). These two solutions of the NLSE (1.1) are connected via the scaling relation (1.6) by choosing the scaling factor $q = 2\alpha_3 > 0$.

In the particular case $a_1 = a_2 = a_3 = \frac{1}{2}$ by using equations (3.3)–(3.7) we finally obtain the following rational (algebraic) solitons:

$$\psi'(x, t) = \frac{\sqrt{2}}{(1 + 4x^2 + 2\epsilon t^2)} \left[\left(\frac{3}{2} - 2x^2 - \epsilon t^2 \right) + 4\mathrm{i}x \right] \mathrm{e}^{\mathrm{i}x} \tag{3.6}$$

where $\epsilon = 1$ corresponds to the regular rational (algebraic) bright soliton (see figure 3) and $\epsilon = -1$ corresponds to the singular rational (algebraic) dark soliton.

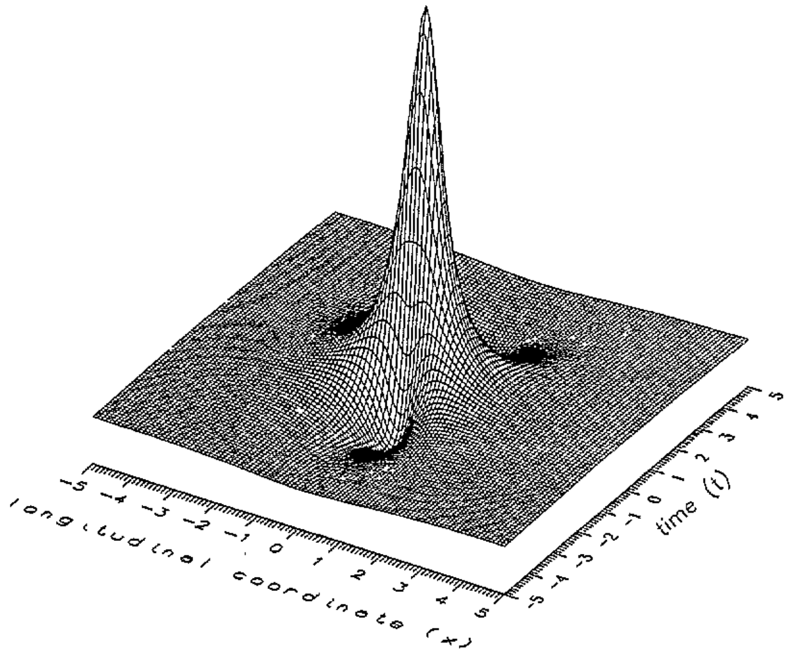

Figure 3. Intensity profile $|\psi'|^2$ versus longitudinal coordinate x and time t.

We note that these solutions correspond to finite boundary conditions at $t = \pm\infty$, i.e., $\psi' \to -(1/\sqrt{2})e^{ix}$ as $|t| \to \infty$.

3.3. $Q_4 = Q_3 < Q_2 = Q_1$

This case is obtained for $\alpha_1 = \alpha_2 = 0$, $\alpha_3 > 0$ and $z = 0$ (see figure 1(c)). Here the roots Q_i are symmetric with respect to the origin: $Q_1 = Q_2 = \alpha_3^{1/2}$, $Q_3 = Q_4 = -\alpha_3^{1/2}$. This case is realized only for $\epsilon = -1$.

The solution Q on the branches (1) in figure 1(c) is:

$$Q = \alpha_3^{1/2} \operatorname{cotanh}(\alpha_3^{1/2} t) \tag{3.7}$$

and for the function $\varphi(x)$ we have the simple expression $\varphi(x) = 2\alpha_3 x$. Thus the function $\psi(x, t)$ is the singular solution:

$$\psi(x, t) = \alpha_3^{1/2} \operatorname{cotanh}(\alpha_3^{1/2} t) \exp(2i\alpha_3 x). \tag{3.8}$$

The corresponding solution on the branch (2) in figure 1(c) is:

$$Q = \alpha_3^{1/2} \tanh(\alpha_3^{1/2} t) \tag{3.9}$$

the function $\varphi(x)$ remaining unmodified. Thus with $q = \alpha_3^{1/2}$ we obtain the dark soliton solution (1.8).

3.4. $\alpha_1 = \alpha_2 > 0$, $\alpha_3 > \alpha_1$ *and* $0 \leqslant z \leqslant \alpha_1$ (see figures 1(*d*) and 2(*b*))

Here we have $Q_1 = 2(\alpha_1 - z)^{1/2} + (\alpha_3 - z)^{1/2}$, $Q_2 = -2(\alpha_1 - z)^{1/2} + (\alpha_3 - z)^{1/2}$, and $Q_3 = Q_4 = -(\alpha_3 - z)^{1/2}$. In this case for $\epsilon = -1$ we have:

$$Q = \begin{cases} \dfrac{\alpha_3 - 2\alpha_1 + z + [(\alpha_1 - z)(\alpha_3 - z)]^{1/2}\cosh\beta t}{(\alpha_3 - z)^{1/2} - (\alpha_1 - z)^{1/2}\cosh\beta t} & Q \geqslant Q_1 \text{ or } Q \leqslant Q_3 \\ \dfrac{\alpha_3 - 2\alpha_1 + z - [\alpha_1 - z)(\alpha_3 - z)]^{1/2}\cosh\beta t}{(\alpha_3 - z)^{1/2} + (\alpha_1 - z)^{1/2}\cosh\beta t} & Q_3 \leqslant Q \leqslant Q_2 \end{cases} \tag{3.11}$$

Here $\beta = 2(\alpha_3 - \alpha_1)^{1/2}$, the solution (3.10) corresponds to $Q(x, t = 0) = Q_1$ and the solution (3.11) corresponds to $Q(x, t = 0) = Q_2$.

For $\epsilon = 1$ the solution is:

$$Q = \frac{\alpha_3 - 2\alpha_1 + z \pm [(\alpha_1 - z)(\alpha_3 - z)]^{1/2}\cos\beta t}{(\alpha_3 - z)^{1/2} \mp (\alpha_1 - z)^{1/2}\cos\beta t} \qquad Q_2 \leqslant Q \leqslant Q_1 \tag{3.12}$$

where β is as before and the upper and the lower signs in (3.12) correspond to $Q(x, t = 0) = Q_1$ and $Q(x, t = 0) = Q_2$, respectively.

We note that the solution (3.10) which corresponds to the branches (1) in figure 1(*d*) is singular, the solution (3.11) which corresponds to the branch (2) in figure 1(*d*) represents the dark solitary wave and the solution (3.12) is regular and periodic.

In this case the functions $z(x)$ and $\varphi(x)$ are:

$$z(x) = \frac{\alpha_1\alpha_3 \sinh^2 \mu x}{\alpha_3 \cosh^2 \mu x - \alpha_1} \tag{3.13}$$

where $\mu = 4[\alpha_1(\alpha_3 - \alpha_1)]^{1/2}$ and

$$\varphi(x) = 2\alpha_3 x + \arctan\left[\left(\frac{\alpha_1}{\alpha_3 - \alpha_1}\right)^{1/2} \tanh \mu x\right]. \tag{3.14}$$

In the case $\epsilon = 1$ and for the particular choice $a_1 = a_2 = a$, where $0 \leqslant a \leqslant a_3 = \frac{1}{2}$, we finally obtain from (3.12)–(3.14) the following one-parameter family of solutions $\psi'(x, t)$ which describe the periodic wave patterns that are generated by the self-phase modulational instability:

$$\psi'(x, t) = -\frac{[(1 - 4a)\cosh\mu_0 x \mp (2a)^{1/2}\cos\beta_0 t - \mathrm{i}\mu_0 \sinh\mu_0 x]}{\sqrt{2}[\cosh\mu_0 x \mp (2a)^{1/2}\cos\beta_0 t]}\mathrm{e}^{\mathrm{i}x} \tag{3.15}$$

where

$$\beta_0 = [2(1 - 2a)]^{1/2} \qquad \mu_0 = [8a(1 - 2a)]^{1/2}. \tag{3.16}$$

The solutions with different signs in (3.15) correspond to a shift in the variable t equal to the semiperiod of the modulation: $t \to t + \pi/\beta_0$. In the particular case $a = \frac{1}{4}$ the solution (3.15) becomes [46]:

$$\psi'(x, t) = -\frac{(\cos t \pm \mathrm{i}\sqrt{2}\sinh x)}{\sqrt{2}[\sqrt{2}\cosh x \mp \cos t]}\mathrm{e}^{\mathrm{i}x}. \tag{3.17}$$

If we write $\psi'(x,t) = w(x,t)e^{ix}$, then from (3.17) it results

$$w(x,t) \to -\frac{1}{\sqrt{2}}\exp\left(\mp i\frac{\pi}{2}\right) \text{ as } x \to \pm\infty$$

Thus in the process of evolution from $x = -\infty$ to $x = +\infty$ we observe the phenomenon of the return to the initial amplitude $1/\sqrt{2}$ but the phase is reversed ($\Delta\varphi = \pi$).

The phenomenon of modulational instability occurs through the interplay between self-phase modulation and anomalous group velocity dispersion and manifests itself as the break-up of continuous wave radiation into a periodic sequence of optical pulses. In figures 4–6 we show the periodic wave patterns that are generated by the self-phase temporal modulation instability (two periods are shown).

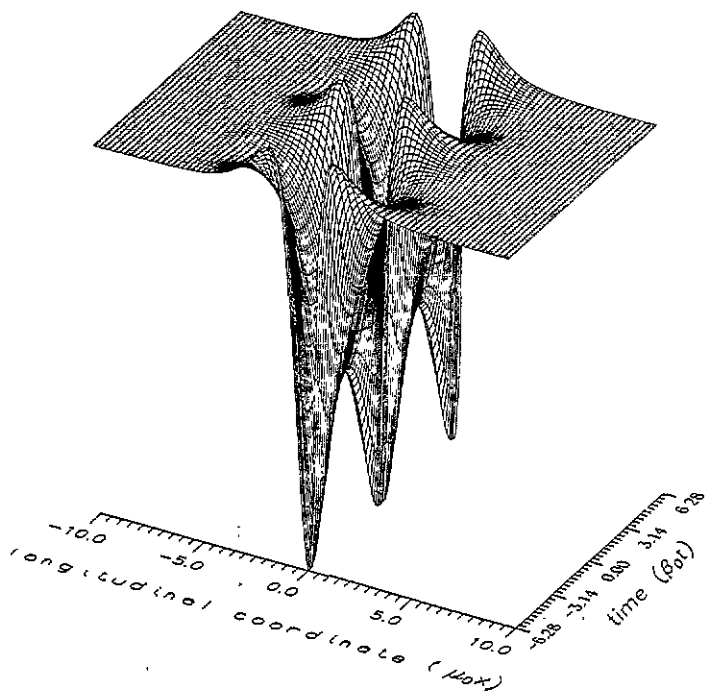

Figure 4. Intensity profile $|\psi'|^2$ versus longitudinal coordinate ($\mu_0 x$) and time ($\beta_0 t$), with $a = 0.125$.

In the case $\epsilon = -1$ and for the particular choice $a_1 = a_2 = a$, where $0 \leqslant a \leqslant a_3 = \frac{1}{2}$ we finally obtain from (3.10)–(3.11) and (3.13)–(3.14) the following one-parameter family of solutions with finite boundary conditions at $t \to \pm\infty$ showing the collision of two dark waves of equal amplitudes:

$$\psi'(x,t) = -\frac{[(1-4a)\cosh\mu_0 x \mp (2a)^{1/2}\cosh\beta_0 t - i\mu_0 \sinh\mu_0 x]}{\sqrt{2}[\cosh\mu_0 x \mp (2a)^{1/2}\cosh\beta_0 t]}e^{ix} \tag{3.18}$$

where β_0 and μ_0 are given by (3.16).

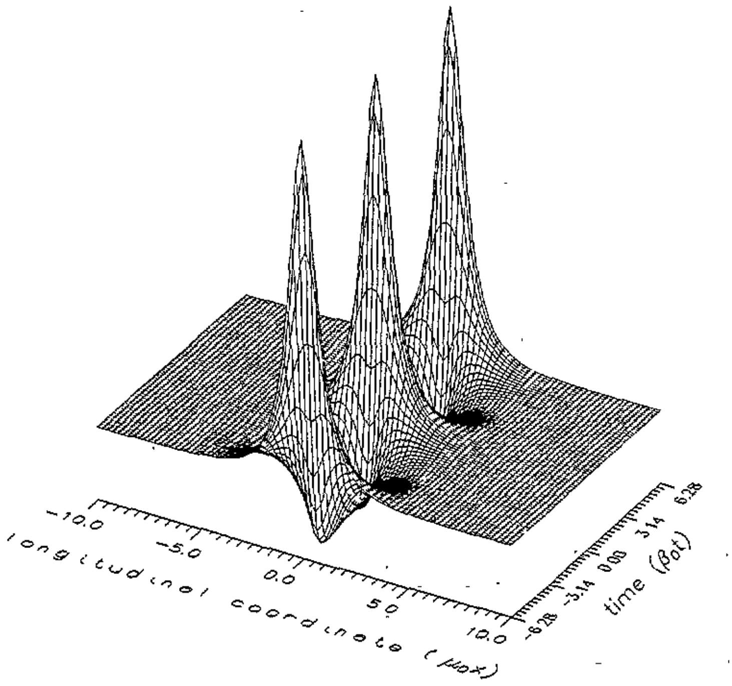

Figure 5. Same as figure 4, with $a = 0.25$.

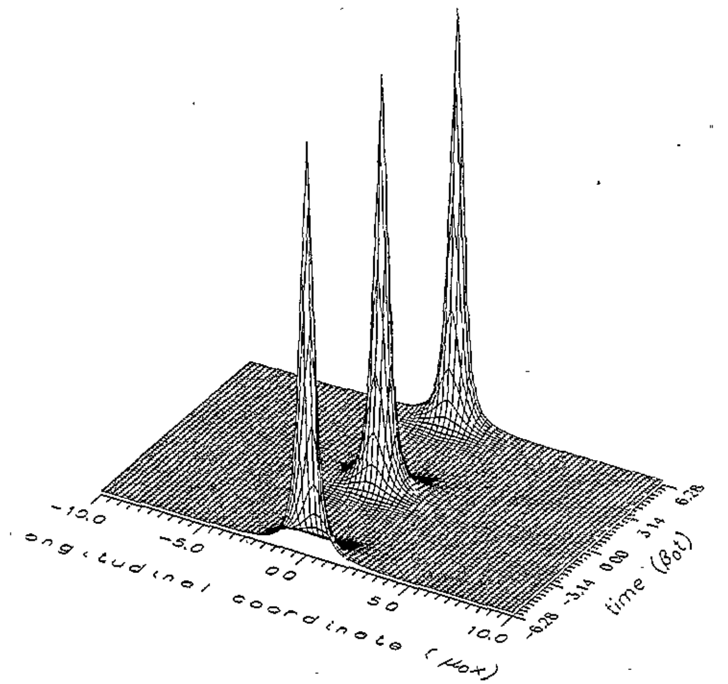

Figure 6. Same as figure 4, with $a = 0.375$.

The solution corresponding to the upper signs in (3.18) is singular and the solution corresponding to the lower signs in (3.18) is regular. In the particular case $a = \frac{1}{4}$ the regular solution (3.18) becomes:

$$\psi'(x,t) = -\frac{(\cosh t - \mathrm{i}\sqrt{2}\sinh x)}{\sqrt{2}[\sqrt{2}\cosh x + \cosh t]}\mathrm{e}^{\mathrm{i}x} \tag{3.19}$$

If we write $\psi'(x,t) = w(x,t)\mathrm{e}^{\mathrm{i}x}$, we thus have

$$w(x,t) \to -\frac{1}{\sqrt{2}}\mathrm{e}^{\mp \mathrm{i}\pi/2} \text{as } x \to \pm\infty$$

therefore in the process of evolution from $x = -\infty$ to $x = +\infty$ the amplitude $1/\sqrt{2}$ is recovered but the phase is reversed ($\Delta\varphi = \pi$). A simple analysis of the expression (3.19) for $\psi'(x,t)$ shows that for $x = 0$ the modulus $|\psi'(x,t)|$ has only one dip at the centre of the pulse, i.e. at $t = 0$ but for every $x \neq 0$ the modulus has two symmetric dips at $t = \pm\tau$, where

$$\tau = \cosh^{-1}\left(\frac{\sqrt{2}\sinh^2 x}{\cosh x}\right).$$

Thus the solution (3.19) describes the splitting of a dark pulse into a pair of shallow grey waves which move apart with equal and opposite transverse components of the velocities as predicted by the inverse scattering theory [33], [4].

Figures 7–8 show the evolution of the intensity profile for the regular solution (3.18) with $a = 0.125$ and $a = 0.375$. We note that the parameter a can be related to the contrast of the separated solitons, which is defined in photometry as

$$C = \frac{I_{\max} - I_{\min}}{I_{\max} + I_{\min}}$$

and gives the visibility of the solitons. In our case we obtain the simple expression $C = a/(1-a)$.

3.5. $\alpha_2 = \alpha_3 > 0$ *and* $0 \leqslant z \leqslant \alpha_1 < \alpha_3$ (see figures 1(*d*) and 2(*c*))

Here we have the following roots: $Q_1 = (\alpha_1 - z)^{1/2} + 2(\alpha_3 - z)^{1/2}$, $Q_2 = Q_3 = -(\alpha_1 - z)^{1/2}$, $Q_4 = (\alpha_1 - z)^{1/2} - 2(\alpha_3 - z)^{1/2}$.

In this case, for $\epsilon = -1$, we obtain the singular periodic solution:

$$Q = \frac{2\alpha_3 - \alpha_1 - z - [(\alpha_1 - z)(\alpha_3 - z)]^{1/2}\cos\beta t}{(\alpha_3 - z)^{1/2}\cos\beta t - (\alpha_1 - z)^{1/2}} \qquad Q \leqslant Q_4 \text{ or } Q \geqslant Q_1 \tag{3.20}$$

and for $\epsilon = 1$ we have the following bright solitary wave:

$$Q = \begin{cases} \dfrac{\alpha_1 - 2\alpha_3 + z - [(\alpha_1 - z)(\alpha_3 - z)]^{1/2}\cosh\beta t}{(\alpha_1 - z)^{1/2} + (\alpha_3 - z)^{1/2}\cosh\beta t} & Q_4 \leqslant Q \leqslant Q_2 \quad (3.21) \\[2ex] \dfrac{2\alpha_3 - \alpha_1 - z - [(\alpha_1 - z)(\alpha_3 - z)]^{1/2}\cosh\beta t}{(\alpha_3 - z)^{1/2}\cosh\beta t - (\alpha_1 - z)^{1/2}} & Q_2 \leqslant Q \leqslant Q_1 \quad (3.22) \end{cases}$$

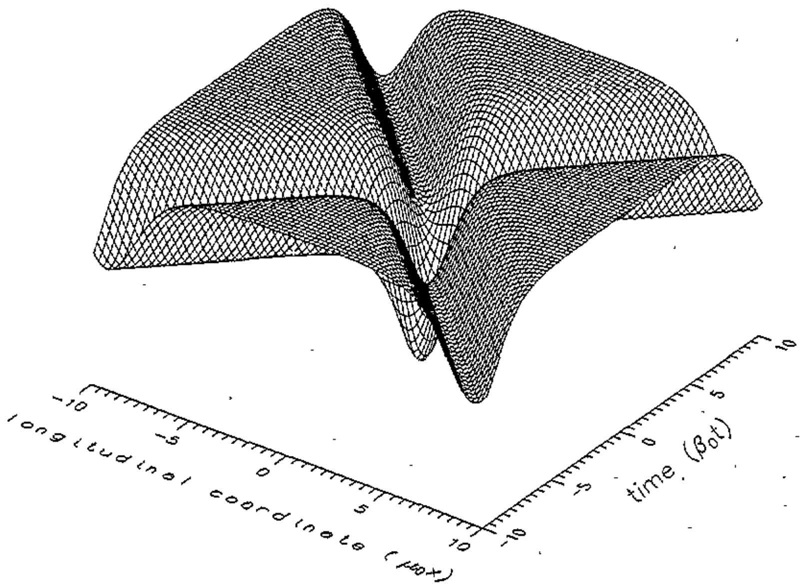

Figure 7. Intensity profile $|\psi'|^2$ versus longitudinal coordinate ($\mu_0 x$) and time ($\beta_0 t$), with $a = 0.125$.

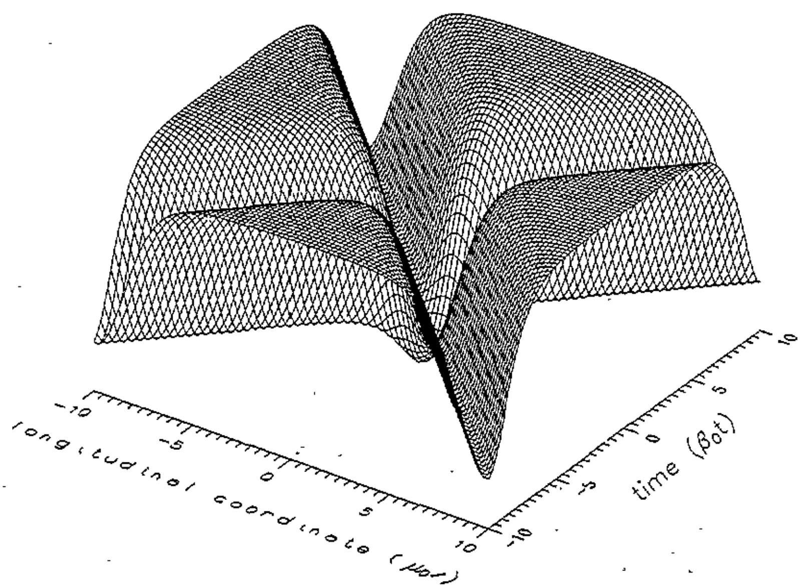

Figure 8. Same as figure 7, with $a = 0.375$.

where $\beta = 2(\alpha_3 - \alpha_1)^{1/2}$. The solution (3.21) corresponds to $Q(x, t = 0) = Q_4$ and the solution (3.22) corresponds to $Q(x, t = 0) = Q_1$.

We note that from (3.21), (3.22) and (2.11), by choosing $\alpha_1 = 0$ and $z = 0$, we obtain the bright soliton solution (1.7) with $q = 2\alpha_3^{1/2}$.

The functions $z(x)$ and $\varphi(x)$ are:

$$z(x) = \frac{\alpha_1\alpha_3 \sin^2 \mu x}{\alpha_3 - \alpha_1 \cos^2 \mu x} \tag{3.23}$$

where $\mu = 4[\alpha_3(\alpha_3 - \alpha_1)]^{1/2}$ and

$$\varphi(x) = 2\alpha_1 x + \arctan\left[\left(\frac{\alpha_3}{\alpha_3 - \alpha_1}\right)^{1/2} \tan \mu x\right]. \tag{3.24}$$

Thus the corresponding solution $\psi(x, t)$ given by (2.23) is periodic in the temporal variable t and double periodic in the spatial variable x.

In the case $\epsilon = 1$ and for the particular choice $a = a_1 \leqslant a_2 = a_3 = \frac{1}{2}$ we find from (3.21)–(3.24) the following one-parameter family of solutions of the NLSE (1) with finite boundary conditions at $t \to \pm\infty$:

$$\psi'(x, t) = \frac{[2(1-a)\cos\rho_0 x \pm (2a)^{1/2}\cosh\beta_0 t + \mathrm{i}\rho_0 \sin\rho_0 x]}{(-2a^{1/2}\cos\rho_0 x \mp 2^{1/2}\cosh\beta_0 t)} \mathrm{e}^{2\mathrm{i}ax} \tag{3.25}$$

where $\rho_0 = 2(1-2a)^{1/2}$ and β_0 is given by (3.16). We see from (3.25) that

$$\psi' \to -a^{1/2}\mathrm{e}^{2\mathrm{i}ax} \text{as } |t| \to \infty$$

i.e., for $|t| \gg 1$ this waveform approaches a continuous wave with amplitude $a^{1/2}$. The solutions with different signs in (3.25) correspond to a shift in the spatial variable x equal to the semiperiod of the modulation: $x \to x + \pi/\rho_0$. We note that the soliton solution (3.25) was first obtained [50] by using the inverse scattering technique for finite boundary conditions at $t = \pm\infty$. For $0 < a < \frac{1}{2}$ the solution (3.25) describes the bright solitons superimposed onto a continuous wave background. The soliton amplitude evolves periodically along the longitudinal direction x with period $(1-2a)^{-1/2}\pi$ [51].

We note also that the bright one-soliton solution of the NLSE (1.1) can be obtained from (3.25) in the limit $a \to 0$:

$$\psi'(x, t) = \mp\sqrt{2}\,\mathrm{sech}(\sqrt{2}t)\exp(2\mathrm{i}x).$$

In figures 9–10 we show the evolution of a bright soliton on a continuous wave background for $a = 0.125$ and $a = 0.375$.

3.6. $\alpha_1 < \alpha_2 < \alpha_3$ and $0 \leqslant z \leqslant \alpha_3$ (see figures 1(*f*) and 2(*d*))

Now we are left with:

$$Q_1 = (\alpha_1 - z)^{1/2} + (\alpha_2 - z)^{1/2} + (\alpha_3 - z)^{1/2} \qquad Q_2 = (\alpha_3 - z)^{1/2} - (\alpha_1 - z)^{1/2} - (\alpha_2 - z)^{1/2}$$

$$Q_3 = (\alpha_2 - z)^{1/2} - (\alpha_1 - z)^{1/2} - (\alpha_3 - z)^{1/2} \qquad Q_4 = (\alpha_1 - z)^{1/2} - (\alpha_2 - z)^{1/2} - (\alpha_3 - z)^{1/2}.$$

For $\epsilon = -1$ the solution Q is:

$$Q = \begin{cases} \dfrac{Q_1(Q_2 - Q_4) - Q_2(Q_1 - Q_4)\mathrm{sn}^2(\beta t, m)}{(Q_2 - Q_4) - (Q_1 - Q_4)\mathrm{sn}^2(\beta t, m)} & Q \geqslant Q_1 \text{ or } Q \leqslant Q_4 \quad (3.26) \\[2ex] \dfrac{Q_2(Q_1 - Q_3) - Q_1(Q_2 - Q_3)\mathrm{sn}^2(\beta t, m)}{(Q_1 - Q_3) - (Q_2 - Q_3)\mathrm{sn}^2(\beta t, m)} & Q_3 \leqslant Q \leqslant Q_2 \quad (3.27) \end{cases}$$

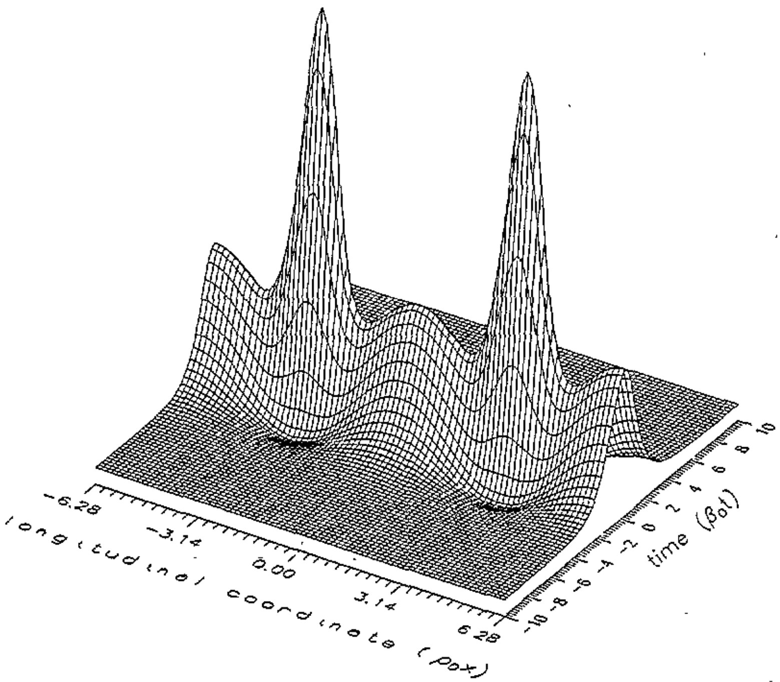

Figure 9. Evolution of a bright soliton on a continuous wave background with $a = 0.125$.

where $\mathrm{sn}(t, m)$ is the Jacobi elliptic function, $\beta = (\alpha_3 - \alpha_1)^{1/2}$ and $m^2 = (\alpha_3 - \alpha_2)/(\alpha_3 - \alpha_1)$. We note that the solution (3.26) which corresponds to the branches (1) in figure 1(f) is singular and periodic and the solution (3.27) which corresponds to the branch (2) in figure 1(f) is regular and periodic.

If $\epsilon = 1$ we are left with the following solution:

$$Q = \begin{cases} \dfrac{Q_4(Q_1 - Q_3) + Q_1(Q_3 - Q_4)\mathrm{sn}^2(\beta t, m)}{(Q_1 - Q_3) + (Q_3 - Q_4)\mathrm{sn}^2(\beta t, m)} & Q_4 \leqslant Q \leqslant Q_3 \qquad (3.28) \\[2ex] \dfrac{Q_2(Q_1 - Q_3) - Q_3(Q_1 - Q_2)\mathrm{sn}^2(\beta t, m)}{(Q_1 - Q_3) - (Q_1 - Q_2)\mathrm{sn}^2(\beta t, m)} & Q_2 \leqslant Q \leqslant Q_1 \qquad (3.29) \end{cases}$$

where $\beta = (\alpha_3 - \alpha_1)^{1/2}$ and $m^2 = (\alpha_2 - \alpha_1)/(\alpha_3 - \alpha_1)$. The solutions (3.21), (3.22) are regular and periodic.

In this case the functions $z(x)$ and $\varphi(x)$ are:

$$z(x) = \frac{\alpha_3 \alpha_1 \mathrm{sn}^2(\mu x, k)}{\alpha_3 - \alpha_1 \mathrm{cn}^2(\mu x, k)} \tag{3.30}$$

where $\mu = 4[\alpha_2(\alpha_3 - \alpha_1)]^{1/2}$, $k^2 = [\alpha_1(\alpha_3 - \alpha_2)]/[\alpha_2(\alpha_3 - \alpha_1)]$ and

$$\varphi(x) = 2(\alpha_1 + \alpha_2 - \alpha_3)x + \frac{4\alpha_3}{\mu}\Pi(n; \mu x, k). \tag{3.31}$$

Here $n = \alpha_1/(\alpha_1 - \alpha_3)$ and $\Pi(n; \mu x, k)$ is the incomplete elliptic integral of the third kind [48]:

$$\Pi(n; \mu x, k) = \int_0^{\mu x} \frac{\mathrm{d}y}{1 - n\,\mathrm{sn}^2(y, k)}. \tag{3.32}$$

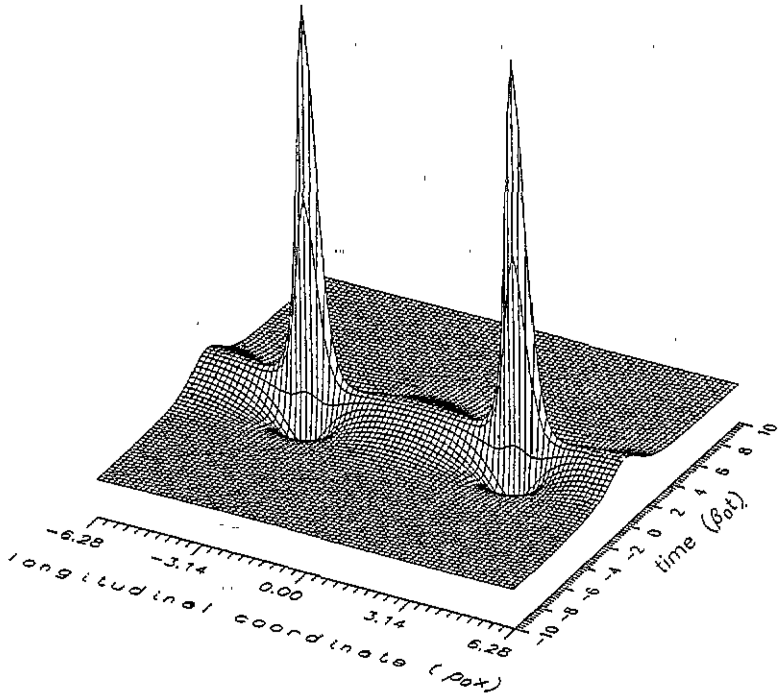

Figure 10. Same as figure 9, with $a = 0.375$.

3.7. α_1 and α_2 are complex conjugate numbers and $0 \leqslant z \leqslant \alpha_3$

Suppose $\alpha_1 = \alpha_2^* = \rho + \mathrm{i}\eta$ (see figures 1(g) and 2(e)) then we have the roots: $Q_{1,2} = -b \pm d$, $Q_{3,4} = b \pm \mathrm{i}c$ where

$$b = (\alpha_3 - z)^{1/2};\, d, c = \{2[(\rho - z)^2 + \eta^2]^{1/2} \pm 2(\rho - z)\}^{1/2}.$$

The solution Q for $\epsilon = -1$ is:

$$Q = \frac{Q_1 B - Q_2 A + (Q_1 B + Q_2 A)\mathrm{cn}(pt, m)}{B - A + (A + B)\mathrm{cn}(pt, m)} \qquad Q \leqslant Q_2 \text{ or } Q \geqslant Q_1 \tag{3.33}$$

where $p = (AB)^{1/2}$, $m^2 = [(A + B)^2 - (Q_1 - Q_2)^2]/(4AB)$

$$A^2 = (d - 2b)^2 + c^2 \qquad B^2 = (d + 2b)^2 + c^2$$

and for $\epsilon = 1$ becomes:

$$Q = \frac{Q_1 B + Q_2 A + (Q_2 A - Q_1 B)\mathrm{cn}(pt, m)}{A + B + (A - B)\mathrm{cn}(pt, m)} \qquad Q_2 \leqslant Q \leqslant Q_1 \tag{3.34}$$

with A, B and m^2 as before with the change $B \Leftrightarrow -B$.

In this case the function $z(x)$ is:

$$z(x) = \frac{\alpha_3(1 - \nu)[1 + \mathrm{cn}(2\mu x, k)]}{2[1 - \nu\, \mathrm{cn}(2\mu x, k)]} \tag{3.35}$$

where

$$\nu = \frac{f-g}{f+g} \qquad \mu = 4(fg)^{1/2}$$

$$f = [(\alpha_3-\rho)^2+\eta^2]^{1/2} \qquad g = (\rho^2+\eta^2)^{1/2}$$

$$k^2 = \frac{1}{2}\left[1 - \frac{\eta^2+\rho(\rho-\alpha_3)}{fg}\right].$$

From (2.11) and (3.35) we finally find the expression for $\varphi(x)$ which is valid for both $\epsilon = 1$ and $\epsilon = -1$.

$$\varphi(x) = 2(2\rho+\alpha_3)x + \frac{4g}{\mu}[(1-n_1)\Pi(n_1;\mu x,k) + (n_2-1)\Pi(n_2;\mu x,k)] \tag{3.36}$$

where

$$n_1 = \frac{2fk^2}{f-g+\alpha_3} \qquad \text{and} \qquad n_2 = \frac{2fk^2}{f-g-\alpha_3}.$$

The explicit analytical solutions (3.33)–(3.36) are periodic in x and t so we obtain the result that the solution $\psi(x,t)$ is double periodic with respect to the spatial variable x and periodic with respect to the time variable t. We note that the solution (3.33) is singular and the solution (3.34) is finite.

3.8. $\alpha_1 < \alpha_2 \leqslant z \leqslant \alpha_3$ and $z \geqslant 0$

This case is realized only for $\epsilon = -1$ and completes the list of possible solutions (see figure 1(*h*)). Then we have the roots Q_i:

$$Q_{1,2} = (\alpha_3-z)^{1/2} \pm \mathrm{i}[(z-\alpha_1)^{1/2} - (z-\alpha_2)^{1/2}] = b_1 \pm \mathrm{i}a_1$$

$$Q_{3,4} = -(\alpha_3-z)^{1/2} \pm \mathrm{i}[(z-\alpha_1)^{1/2} + (z-\alpha_2)^{1/2}] = b_2 \pm \mathrm{i}a_2.$$

The solution Q is:

$$Q = \frac{b_1 - a_1g_1 + (a_1+b_1g_1)\mathrm{sc}(pt,m)}{1+g_1\mathrm{sc}(pt,m)} \tag{3.37}$$

where

$$\mathrm{sc}(t,m) = \frac{\mathrm{sn}(t,m)}{\mathrm{cn}(t,m)} \qquad p = \frac{(A+B)}{2} \qquad m^2 = \frac{4AB}{(A+B)^2}$$

$$A^2 = (b_1-b_2)^2 + (a_1+a_2)^2 \qquad B^2 = (b_1-b_2)^2 + (a_1-a_2)^2$$

$$g_1^2 = \frac{4a_1^2-(A-B)^2}{(A+B)^2-4a_1^2}.$$

Thus for this choice of parameters α_i we are left also with the singular periodic solution. Finally we have the following expression for the functions $z(x)$ and $\varphi(x)$:

$$z(x) = \frac{\alpha_2(\alpha_3-\alpha_1) - \alpha_1(\alpha_3-\alpha_2)\mathrm{sn}^2(\mu x,k)}{(\alpha_3-\alpha_1) - (\alpha_3-\alpha_1)\mathrm{sn}^2(\mu x,k)} \tag{3.38}$$

where $\mu = 4[\alpha_2(\alpha_3 - \alpha_1)]^{1/2}$, $k^2 = [\alpha_1(\alpha_3 - \alpha_2)]/[\alpha_2(\alpha_3 - \alpha_1)]$ and

$$\varphi(x) = 2(\alpha_1 + \alpha_2 + \alpha_3)x - 4\mu\alpha_1 x + 4(\alpha_1 - \alpha_2)\Pi(n; \mu x, k) \tag{3.39}$$

with $n = (\alpha_3 - \alpha_2)/(\alpha_3 - \alpha_1)$.

Finally we mention that on the branch between two double roots (see the case (c) in figure 1 when $Q_2 = Q_1 = -Q_4 = -Q_3$) there exists the dark soliton solution. If the two double roots are asymmetric with respect to origin the solution becomes the kink-like wave. By using the method developed in our paper we cannot find the kink-like solution because it is not possible to have $Q_2 = Q_1 \neq -Q_4 = -Q_3$. The solitary wave occurs on the branch between a double root and a simple root and the regular periodic solution occurs on the branch between two simple roots.

4. Conclusions

The method developed in this paper allows us to obtain a class of general solutions of NLSE describing the propagation of picoseconds light pulses in optical fibres in both the normal and anomalous group velocity dispersion regime. We found exact analytical solutions for the nonlinear wave equation (1.1) with $\epsilon = \pm 1$ such as the bright or dark solitary waves, bright or dark soliton solutions, rational (algebraic) bright or dark solitons, periodic and stationary solutions. These solutions are obtained by a direct method which is based on the relationship (1.4) between the real and imaginary parts of the complex amplitude $\psi(x, t) = u(x, t) + \mathrm{i}v(x, t)$.

From the general exact solutions we obtain as particular cases solutions which describe the development of the temporal self-phase modulation instability, the periodic evolution of a bright soliton superimposed on a continuous wave background and the collision of two dark waves with equal amplitudes.

Finally we note that in order to obtain other classes of solutions of the NLSE (1.1) (the higher-order solutions), instead of taking the linear relationship (1.4) between the real and imaginary parts $u(x, t)$ and $v(x, t)$ of the unknown function $\psi(x, t)$, one can choose a rather arbitrary relationship of the form $P(u(x, t), v(x, t), x) = 0$ which is the equation of an algebraic curve in the space of pairs of functions (u, v) whose parameters depend only on the spatial variable x.

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